Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 05–10

UNCERTAINTY PRINCIPLES FOR THE q-HANKEL TRANSFORM

LUIS DANIEL ABREU

ABSTRACT: We prove two propositions related to the support of functions and their q-Hankel transform. The first says that if a function f and its q-Hankel transform both vanish at the points q^{-n}, $n = 1, 2, ...$ then f must vanish identically. The second asserts that if f is supported at $[0, T]$ and its q-Hankel transform at $[0, \Omega]$ then $\Omega T \geq (q; q)_\infty^2$.

KEYWORDS: uncertainty principles; q-Bessel functions; q-Hankel transform.
AMS SUBJECT CLASSIFICATION (2000): 44A15, 33D15.

1. Introduction

The Fourier transform of a $L^1(\mathbf{R})$ function supported on a finite interval (a, b)

$$f^{\wedge}(\omega) = \int_a^b f(t)e^{-\omega it}dt \tag{1}$$

defines an entire function. Therefore, if $f^{\wedge}$ itself has compact support, then it must vanish identically since it vanishes on a set with an accumulation point. By Fourier inversion f itself must vanish identically. This is the most simple manifestation of the uncertainty principle of Fourier analysis which says, in general, that a function and its transform cannot be simultaneously small. The present note pretends to address the question of how to prove such a statement if, instead of the Lebesgue measure, one is working with a measure without an accumulation point outside the interval (a, b).

Consider a number q in the real interval $(0, 1)$. The prototype of the situation just described is the discrete Jackson q-integral

$$\int_0^\infty f(t)\, d_qt = (1-q) \sum_{n=-\infty}^{\infty} f(q^n)\, q^n. \tag{2}$$

where the spectrum of the measure is $\{q^n\}_{n=-\infty}^{\infty}$ which has zero as the only accumulation point. Using the q-integral and a suitable chosen q-analogue of

Received June 2, 2005.
Partial financial assistance by Centro de Matemática da Universidade de Coimbra.

the Bessel function (which we will define in the next section), Koornwinder and Swarttouw defined in [5] a q-analogue of the Hankel transform, $H_q^{\nu}f$, setting

$$\left(H_q^{\nu}f\right)(x)=\int_0^{\infty}(xt)^{\frac{1}{2}}\,J_{\nu}^{(3)}\left(xt;q^2\right)f(t)\,d_qt \tag{3}$$

For the transform $H_q^{\nu}f$, we will prove that, in a convenient normalized space, if f and $H_q^{\nu}f$ vanish at all the points of the spectrum outside the interval $(0,1)$, then f must vanish in the equivalent classes of the normalized space considered. The presentation is organized as follows. In the next section we introduce the notions about q-calculus to be used in the remaining of the paper. In the third section we prove our main theorem and deduce from it a proposition about uniqueness sets in a certain Hilbert space of entire functions. In the last section we obtain some estimates on the kernel of the integral transform and use them to conclude, from a general proposition due to de Jeu [6], that the length of the support of f times the length of the support of $H_q^{\nu}f$ must be bigger than a certain positive quantity, paralleling a classical result about Fourier transforms.

2. Basic definitions and facts

The third Jackson q-Bessel function or the Hahn-Exton q-Bessel function is defined by

$$J_{\nu}(z;q)=\frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}}\sum_{n=0}^{\infty}(-1)^n\,\frac{q^{n(n+1)/2}}{(q^{\nu+1};q)_n(q;q)_n}z^{2n+\nu} \tag{4}$$

The notation $J_{\nu}^{(3)}(z;q)$ is used to distinguish it from the other two known q-Bessel functions. Since this is the only Bessel function appearing on the text, we will drop the superscript for shortness of the notations and write $J_{\nu}(z;q)=J_{\nu}^{(3)}(z;q)$. The symbols in the above definitions are

$$(a;q)_n=(1-q)(1-aq)\ldots\left(1-aq^{n-1}\right) \tag{5}$$

with the zero and infinite cases as

$$(a;q)_0=1 \tag{6}$$

$$(a;q)_{\infty}=\lim_{n\to\infty}(a;q)_n=\prod_{k=0}^{\infty}(1-aq^n) \tag{7}$$

The infinite product above can be written in series form by means of the the Euler formula:

$$(z;q)_{\infty}=\sum_{n=0}^{\infty}(-1)^{n}\frac{q^{\frac{n(n-1)}{2}}}{(q;q)_{n}}x^{n} \tag{8}$$

The $q-$integral in the finite interval $(0,a)$ is

$$\int_{0}^{a}f(t)\,d_{q}t=(1-q)\,a\sum_{n=0}^{\infty}f(aq^{n})\,q^{n} \tag{9}$$

and in the interval $(0,\infty)$

$$\int_{0}^{\infty}f(t)\,d_{q}t=(1-q)\sum_{n=-\infty}^{\infty}f(q^{n})\,q^{n} \tag{10}$$

We will denote by $L_{q}^{p}(X)$ the Banach space induced by the norm

$$\|f\|_{p}=\left[\int_{X}|f(t)|^{p}\,d_{q}t\right]^{\frac{1}{p}}. \tag{11}$$

For entire indices, the functions $J_{n}(x;q)$ are generated by the relation, valid for $|xt|<1$,

$$\frac{\left(qxt^{-1};q\right)_{\infty}}{(xt;q)_{\infty}}=\sum_{n=-\infty}^{\infty}J_{n}(x;q)t^{n} \tag{12}$$

It was shown in [5] that the q-Hankel transform satisfies the inversion formula

$$f(t)=\int_{0}^{\infty}(xt)^{\frac{1}{2}}\left(H_{q}^{\nu}f\right)(x)\,J_{\nu}\left(xt;q^{2}\right)d_{q}x=\left(H_{q}^{\nu}\left(H_{q}^{\nu}f\right)\right)(t) \tag{13}$$

where t takes the values $q^{k},k\in Z$.

3. A vanishing theorem for the q-Hankel transform

The main tool in the proof of the main result in this section is the following completeness criterion, derived in [2] as a consequence of the Phragmén-Lindelöf principle for functions of order less than one.

Theorem A. *Let f and g be defined by their power series expansions as $f(z)=\sum_{n=0}^{\infty}(-1)^{n}a_{n}z^{2n}$ and $g(z)=\sum_{n=0}^{\infty}(-1)^{n}b_{n}z^{2n}$. Denote by λ_{n} the nth zero of g. If the order of f is less than one, then the sequence $\{f(\lambda_{n}x)\}$ is complete $L_{q}(0,1)$ if, as $n\to\infty$,*

$$\frac{a_{n}}{b_{n}}\to 0 \tag{14}$$

Now we formulate and proof our main result, which is an uncertainty principle of a qualitative nature. Essentially it says that a $L_q^1(\mathbf{R}^+)$ function and its q-Hankel transform cannot be both simultaneously supported inside the interval $(0,1)$.

Theorem 1. *Let $f \in L_q^1(\mathbf{R}^+)$ such that both f and its q-Hankel transform vanish at the points $q^{-n}, n \in \mathbf{N}_0$, then*

$$f(q^k) = 0, k \in \mathbf{Z}. \tag{15}$$

that is, $f \equiv 0$ almost everywhere in $L_q^1(\mathbf{R}^+)$. If f is analytic then f must vanish identically in the whole complex plane.

Proof. Let $f \in L_q^1(\mathbf{R}^+)$. If $f(q^{-n}) = 0, n \in \mathbf{N}_0$, then the q-Hankel transform of f is

$$H_q^\nu f(\omega) = \int_0^1 (\omega t)^{\frac{1}{2}} J_\nu\left(\omega t; q^2\right) f(t)\, d_q t. \tag{16}$$

Our second assumption says that

$$(H_q^\nu f)(q^{-n}) = 0, n \in \mathbf{N}_0 \tag{17}$$

therefore, setting $\omega = q^{-n}$ in (16) gives

$$\int_0^1 \left(q^{-n} t\right)^{\frac{1}{2}} J_\nu\left(q^{-n} t; q^2\right) f(t) d_q t = 0, n \in \mathbf{N}_0 \tag{18}$$

Now, in the set up of Theorem A take $f(z) = J_\nu\left(z; q^2\right)$ and $g(z) = (z^2; q^2)_\infty$. Using (4) and (8) together with the trivial observation that $\{q^{-n}\}$ is the sequence of zeros of g gives that, if $\nu > -1$, the sequence $\{J_\nu\left(q^{-n} x; q^2\right)\}$ is complete in $L_q^1(0,1)$. This, together with (18) implies that $f \equiv 0$ in $L_q^1(0,1)$, that is,

$$f(q^n) = 0, n \in \mathbf{N}_0 \tag{19}$$

Combining this with the assumption $f(q^{-n}) = 0, n \in \mathbf{N}_0$ gives

$$f(q^k) = 0, k \in \mathbf{Z} \tag{20}$$

This proves that $f \equiv 0$ almost everywhere in $L_q^1(\mathbf{R}^+)$. Since the set $\{q^k, k \in \mathbf{Z}\}$ has an accumulation point, if f is analytic then it must be the null function. $\square$

Following [1] we introduce the space

$$PW_q^\nu = \left\{ f \in L_q^2\left(\mathbf{R}^+\right) : f\left(x\right) = \int_0^1 \left(tx\right)^{\frac{1}{2}} J_\nu\left(xt; q^2\right) u\left(t\right) d_q t, u \in L_q^2\left(0,1\right) \right\} \tag{21}$$

This can be interpreted as a q-Bessel version of the Paley Wiener space of bandlimited functions. Clearly, PW_q^ν is a Hilbert space of analytic functions. Observe also that, if $\left(H_q^\nu f\right)\left(q^{-n}\right) = 0, n \in \mathbf{N}$, then taking into account definitions (9) and (2), $f = \left(H_q^\nu\left(H_q^\nu f\right)\right)$ is of the form required in (21). Using these concepts, we have the following consequence of the vanishing theorem:

Corollary 1. $\Gamma = \{q^{-n}, n \in \mathbf{N}\}$ *is a set of uniqueness for the space* PW_q^ν.

Proof. Take $f \in PW_q^\nu$ such that $f(q^{-n}) = 0, n \in \mathbf{N}$. If f is of the form required in (21) then $f = H_q^\nu u^*$ where $u^* \in L_q^2\left(\mathbf{R}^+\right)$ is obtained from $u \in L_q^2\left(0,1\right)$ by prescribing $u(q^{-n}) = 0, n \in \mathbf{N}$. By the inversion formula (13), $u^* = H_q^\nu f$. We conclude that $H_q^\nu f(q^{-n}) = 0, n \in \mathbf{N}$. By Theorem 1, $f \equiv 0$. $\square$

Remark 1. *Observe that we proved the following characterization of* PW_q^ν*:*

$$PW_q^\nu = \left\{ f \in L_q^2\left(\mathbf{R}^+\right) : \left(H_q^\nu f\right)\left(q^{-n}\right) = 0, n \in \mathbf{N} \right\} \tag{22}$$

The property $\left(H_q^\nu f\right)\left(q^{-n}\right) = 0, n \in \mathbf{N}$ *can thus be seen as a sort of "q-Hankel-bandlimitedness". It was shown in* [1] *that there are many features in this space analogous to the classical Paley Wiener space, including a sampling theorem and a reproducing kernel.*

4. An uncertainty principle

With the purpose of extending the Donoho and Stark uncertainty principle [3] to an abstract setting, de Jeu [6] obtained a very general proposition, from which we just quote a special case.

Theorem B *If there is a Plancherel theorem for the integral transform in* $L^2(X)$ *whose kernel is* $K(x,t)$*, then, if the support of* f *is* T *and the support of* $(Kf)(x) = \int_X K(x,t) f(t) d\mu(t)$ *is* Ω*, the following inequality holds:*

$$\left\| \mathbf{1}_{T\times\Omega} K(x,t) \right\|_{L^2(\mu,X)\times L^2(\mu,X)} \geq 1 \tag{23}$$

In order to use Theorem B to extract more valuable information about the size of the supports in our study of the q-Hankel transform, we must first obtain bounds for its kernel.

Lemma 1. *If $\nu \geq 0$ and $|x| < q^{-\frac{1}{2}}$, the inequality holds:*

$$|J_\nu(x;q)| \leq \frac{1}{(q;q)_\infty} \tag{24}$$

Proof. If $\nu > 0$, $y > -\frac{1}{2}$ and $x \in \mathbf{R}$, the following q-analogue of the Sonine integral was proved in [1]:

$$\frac{(q;q)_\infty}{(q^\nu;q)_\infty} x^{-\nu} J_{y+\nu}(x;q) = \int_0^1 t^{\frac{y}{2}} \frac{(tq;q)_\infty}{(tq^\nu;q)_\infty} J_y(xt^{\frac{1}{2}};q) d_q t \tag{25}$$

Setting $y = 0$ in (25) and taking absolute values gives

$$|J_\nu(x;q)| \leq \left| x^\nu \frac{(q^\nu;q)_\infty}{(q;q)_\infty} \right| \int_0^1 \left| \frac{(tq;q)_\infty}{(tq^\nu;q)_\infty} J_0(xt^{\frac{1}{2}};q) \right| d_q t \tag{26}$$

We need to estimate the integrand in (25). For the infinite product, observe that if $0 < t < 1$, then

$$\frac{(tq;q)_\infty}{(tq^\nu;q)_\infty} < \frac{1}{(q^\nu;q)_\infty} \tag{27}$$

Now we will show that, if $t < 1$ and $|x| < q^{-\frac{1}{2}}$ then

$$\left| J_0(xt^{\frac{1}{2}};q) \right| \leq 1 \tag{28}$$

This can be seen using a generating function argument as follows. Substituting t by $t^{-1}q$ in (12) and multiplying the two resulting identities gives, if $|xq| < |t| < |x|^{-1}$ (which holds if $|x| < q^{-\frac{1}{2}}$ and $|xt| < 1$)

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} t^{n-m} q^m J_n(x;q) J_m(x;q) = 1 \tag{29}$$

Equating coefficients of t^0 in (29) reveals that, if $|x| < q^{-\frac{1}{2}}$, $\sum_{k=-\infty}^{\infty} q^k \left[J_k(x;q)\right]^2 = 1$. In particular,

$$|J_k(x;q)| \leq q^{-\frac{k}{2}},\ k = 0, 1, ... \tag{30}$$

Now, if $t < 1$ and $|x| < q^{-\frac{1}{2}}$ we also have $|xt| < q^{-\frac{1}{2}}$. Setting $k = 0$ in (30) gives(28). Using this estimates in (26) together with (27) gives

$$|J_\nu(x;q)| \leq \left| x^\nu \frac{1}{(q;q)_\infty} \right| . \tag{31}$$

This proves the lemma. $\square$

We can now state a proposition providing information of a quantitative nature about the supports of f and $H_q^\nu f$.

Theorem 2. *Suppose that $\nu \geq 0$. If the support of f is contained in $[0,T]$ and the support of $H_q^\nu f$ is contained in $[0,\Omega]$, then*

$$\Omega T \geq (q;q)_\infty^2 \tag{32}$$

Proof. First observe that if $\Omega T \geq 1$ then the proposition is trivial, since $(q;q)_\infty < 1$. Thus we can assume without loss of generalization that $\Omega T < 1$. In this case we have $|xt| < 1$ in the square $[0,T] \times [0,\Omega]$ and the use of (24) together with the definition of the q-integral gives

$$\left\| \mathbf{1}_{T\times\Omega}(x,t)\,(xt)^{\frac{1}{2}}\,J_\nu\left(xt;q^2\right)\right\|_{L_q^2(X)\times L_q^2(X)} = \int_0^\Omega \left[\int_0^T \left[(tx)^{\frac{1}{2}}\,J_\nu\left(xt;q^2\right)\right]^2 d_qt\right] d_qx \tag{33}$$

$$\leq \int_0^\Omega \int_0^T \left[\frac{1}{(q;q)_\infty}\right]^2 d_qt d_qx = \frac{\Omega T}{(q;q)_\infty^2} \tag{34}$$

now observe that applying Theorem B to the q-Hankel transform gives

$$1 \leq \left\| \mathbf{1}_{T\times\Omega}\,(xt)^{\frac{1}{2}}\,J_\nu\left(xt;q^2\right)\right\|_{L_q^2(X)\times L_q^2(X)} \tag{35}$$

and the result is proved. $\square$

References

[1] L. D. Abreu, *A q-Sampling Theorem related to the q-Hankel transform,* Proc. Amer. Math. Soc. 133 (2005), 1197-1203.

[2] L. D. Abreu, *Completeness criteria for sequences of special functions,* preprint.

[3] D. L. Donoho, P. B. Stark, *Uncertainty principles and signal recovery.* SIAM J. Appl. Math. 49 (1989), no. 3, 906–931

[4] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge.

[5] T. H. Koornwinder, R. F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms.* Trans. Amer. Math. Soc. 333 (1992), no. 1, 445–461.

[6] M. F. E. de Jeu, *An uncertainty principle for integral operators.* J. Funct. Anal. 122 (1994), no. 1, 247–253.

Luis Daniel Abreu
Departamento de Matemática da Universidade de Coimbra
E-mail address: daniel@mat.uc.pt